Acta Mathematica Sinica, English Series
Oct., 2000, Vol. 16, No.4, pp. 613–626

Acta Mathematica Sinica, English Series

A Besov Estimate for Multilinear Singular Integrals

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Abstract The main purpose of this paper is to prove a good λ inequality for a multilinear singular integral in $\mathbb{R}^n$.

Keywords Multilinear operator, Besov space, Good λ inequality
1991MR Subject Classification 42B20, 42B25

1 Introduction

A well-known result of Coifman, Rochberg and Weiss [1] states that the commutator operator $C_f(g) = T(f \cdot g) - f \cdot T(g)$ (where T is a Calderón Zygmund singular integral operator) is bounded on some $L^p(\mathbb{R}^n), 1 < p < \infty$, if and only if $f \in \text{BMO}$. There are other links between the boundedness properties of the operator C_f and the smoothness of f. A particular case of Janson's [2] result states that $C_f : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded, $1 < p < q < \infty$, if and only if $f \in \text{Lip}\beta$, $\beta = n(1/p - 1/q)$. Here, $\text{Lip}\beta$ is the homogeneous Lipschitz space determined by the first difference operator. The operators considered here are the maximal and limiting operators of the singular integral

$$C_\epsilon(a, b; f)(x) = \int_{|x-y|>\epsilon} P_{m_1}(a; x, y) P_{m_2}(b; x, y) \frac{\Omega(x-y)}{|x-y|^{n+M-2}} f(y) dy, \tag{1}$$

where $P_m(a; x, y) = a(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!} a^{(\alpha)}(y)(x-y)^\alpha$ and $M = m_1 + m_2$, Ω satisfies certain homogeneity, smoothness and symmetry conditions. The BMO estimate for these operators has been obtained by Cohen and Gosselin (see [3]). This paper aims to prove a good λ inequality

Received September 30, 1999, Accepted January 14, 2000
The research is supported by Beijing Education Commission Foundation and Beijing Natural Science Foundation (Project No. 1982005)

for the maximal function $C_*(a, b; f)(x) = \sup_{\epsilon>0} |C_\epsilon(a, b; f)(x)|$ if $a^{(\alpha)}$, $|\alpha| = m_1 - 1$ and $b^{(\eta)}$, $|\eta| = m_2 - 1$ belong to the Besov-Lipschitz class $\dot{\Lambda}_\beta$. As a direct consequence, the limiting operator maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $1/q = 1/p - 2\beta/n$.

2 Preliminaries

Throughout this paper we will work in Euclidean space $\mathbb{R}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the order of α. If b is a smooth function on $\mathbb{R}^n$, $b^{(\alpha)}$ will denote the partial derivative $\frac{\partial^{|\alpha|} b}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$. Let $P_m(b; x, y)$ denote the m-th order Taylor series remainder of b at x expanded about y. More precisely,

$$P_m(b; x, y) = b(x) - \sum_{|\alpha|<m} \frac{b^{(\alpha)}(y)}{\alpha!}(x-y)^\alpha, \tag{2}$$

where $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$ and $(x-y)^\alpha = (x_1-y_1)^{\alpha_1} \cdots (x_n-y_n)^{\alpha_n}$. Let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. In this paper Q will denote a cube with sides parallel to the axes and if b is an integrable function on Q, $m_Q(b)$ will denote the average of b over Q i.e., $|Q|^{-1} \int_Q b(x)dx$. Let $Mf(x)$ denote the Hardy Littlewood maximal function of f and $M_r f(x) = (M(|f|^r)(x))^{1/r}$. We also need a variant version of the maximal function $M_{\alpha,p}(f)(x)$ which was introduced by Muckenhoupt and Wheeden in [4] and defined by

$$M_{\alpha,p}(f)(x) = \sup_{x\in Q} \Big(\frac{1}{|Q|^{1-\alpha p/n}} \int_Q |f(y)|^p dy \Big)^{1/p}.$$

Chanillo [5] proved the following:

Lemma 2.1 *Let $p_1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Then*

$$\|M_{\alpha,p_1}(f)\|_q \le C\|f\|_p. \tag{3}$$

For $\beta > 0$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x,h\in\mathbb{R}^n \\ h\neq 0}} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the k-th difference operator. $\dot{F}_p^{\beta,\infty}$ is the homogeneous Triebel-Lizorkin space. Finally we will let C be a constant that may vary from line to line.

3 Statements of Results

Let Ω be homogeneous of degree zero, satisfy $|\Omega(x) - \Omega(y)| \le C|x-y|$ for $|x| = |y| = 1$, and have vanishing moments up to order $M-2$ over the unit sphere in $\mathbb{R}^n$. Suppose a and b are functions with derivatives of order $m_1 - 1$ and $m_2 - 1$, respectively, in $\dot{\Lambda}_\beta$, $0 < \beta < 1$, and we denote $\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta} = \sum_{|\alpha|=m_1-1}\|a^{(\alpha)}\|_{\dot{\Lambda}_\beta}$ and $\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} = \sum_{|\eta|=m_2-1}\|b^{(\eta)}\|_{\dot{\Lambda}_\beta}$. Now we can state our main results.

Theorem 3.1 (Main estimate) *For $1 < r < p < \infty$, there exists $\gamma_0 > 0$ such that for $0 < \gamma < \gamma_0$ and $\lambda > 0$*

$$\Big|\{x \in \mathbb{R}^n : C_*(a, b; f)(x) > 3\lambda, \|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta, p}(f)(x) \le \gamma\lambda\}\Big| \le C\gamma^r\Big|\{x \in \mathbb{R}^n : C_*(a, b; f)(x) > \lambda\}\Big|. \tag{4}$$

Theorem 3.2 *If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then the limiting operator $C(a, b; f)(x) = \lim_{\epsilon\to 0} C_\epsilon(a, b; f)(x)$ exists almost everywhere and satisfies the estimate*

$$\|C(a, b; f)\|_q \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\|f\|_p, \tag{5}$$

when $1/q = 1/p - 2\beta/n$ and $1/p > 2\beta/n$.

Theorem 3.3 *If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $0 < \beta < 1/2$, then*

$$\|C(a, b; f)\|_{\dot{F}_p^{2\beta, \infty}} \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\|f\|_p. \tag{6}$$

4 Some Lemmas and Proof of the Theorems

Lemma 4.1 (See [3]) *Let $b(x)$ be a function on $\mathbb{R}^n$ with m-th order derivatives in $L^q(\mathbb{R}^n)$ where $q > n$. Then*

$$|P_m(b; x, y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m}\Big(\frac{1}{|Q(x, y)|}\int_{Q(x, y)}|b^{(\alpha)}(t)|^q dt\Big)^{1/q}, \tag{7}$$

where $Q(x, y)$ is the cube centered at x with edges parallel to the axes and having diameter $5\sqrt{n}|x-y|$.

Lemma 4.2 (See [6]) (a) *For $0 < \beta < 1$, $1 \le q < \infty$,*

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\Big(\frac{1}{|Q|}\int_Q |f(x) - m_Q(f)|^q dx\Big)^{1/q}.$$

(b) *For $0 < \beta < 1$, $1 < p < \infty$,*

$$\|h\|_{\dot{F}_p^{\beta, \infty}} \approx \Big\|\sup_{\cdot\in Q}\frac{1}{|Q|^{1+\beta/n}}\int_Q |h(x) - m_Q(h)|dx\Big\|_p.$$

Lemma 4.3 (See [7]) *Let $Q^* \subset Q$. Then $|m_{Q^*}(f) - m_Q(f)| \le C\|f\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}$.*

We now turn to the proof of the main estimate (4). Using a Whitney argument we write $\{x \in \mathbb{R}^n : C_*(a, b; f)(x) > \lambda\}$ as a union of cubes $\{Q_j\}$ with mutually disjoint interiors and with distance from each Q_j to $\mathbb{R}^n\setminus\cup_j Q_j$ comparable to the diameter of Q_j. It is now sufficient to prove the main estimate for each Q_j. There exists a constant $C = C(n)$ such that for each j the cube $\tilde{Q}_j$ with the same center as Q_j but with $\mathrm{diam}(\tilde{Q}_j) = C(n)\mathrm{diam}(Q_j)$ intersects $\mathbb{R}^n \setminus \cup_j Q_j$. Thus for each j there exists a point $x_0 = x_0(j) \in \tilde{Q}_j$ such that $C_*(a,\, b;\, f)(x_0) \le \lambda$.

We now fix a cube Q_j and assume that there exists a point $z = z(j)$ with

$$\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta,\, p}(f)(z) \le \gamma\lambda.$$

(If no such point exists, the result is trivial for Q_j). Let $\bar{Q}_j = \tilde{\tilde{Q}}_j$ and write $f = f_1 + f_2$ where $f_1 = f\chi_{\bar{Q}_j}$. We now make appropriate estimates on f_1 and f_2 separately.

The f_1 estimate. We choose a C_0^∞ function φ such that $\varphi(x) \equiv 1$ for $x \in \bar{Q}_j$, $\varphi(x) \equiv 0$ for $x \notin \bar{\bar{Q}}_j$, $|\varphi(x)| \le 1$ for all x, and for any multi-index α with $|\alpha| \le M$, $|\varphi^{(\alpha)}(x)| \le C(\mathrm{diam}(\bar{\bar{Q}}_j))^{-|\alpha|}$. We note that C is independent of j. We now define

$$a_\varphi(y) \equiv P_{m_1-1}\Big(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})(\cdot)^\alpha;\, y,\, z\Big)\varphi(y),$$
$$b_\varphi(y) \equiv P_{m_2-1}\Big(b(\cdot) - \sum_{|\alpha|=m_2-1} \frac{1}{\alpha!} m_{Q_j}(b^{(\alpha)})(\cdot)^\alpha;\, y,\, z\Big)\varphi(y).$$

We note that a_φ and b_φ have support in $\bar{\bar{Q}}_j$. We now estimate the derivatives of order $m_1 - 1$ of a_φ. Let γ be a multi-index of order $m_1 - 1$. Then

$$\begin{aligned} a_\varphi^{(\gamma)}(y) &= \sum_{\gamma=\mu+\nu} C_{\mu,\,\nu}\Big\{P_{m_1-1}^{(\mu)}\Big(a(\cdot) - \sum_{|\alpha|=m_1-1}\frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})(\cdot)^\alpha;\, y,\, z\Big)\varphi^{(\nu)}(y)\Big\} \\ &= \sum_{\gamma=\mu+\nu} C_{\mu,\,\nu} P_{m_1-1-|\mu|}\Big(\frac{\partial^\mu}{\partial y^\mu}\Big(a(\cdot) - \sum_{|\alpha|=m_1-1}\frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})(\cdot)^\alpha\Big);\, y,\, z\Big)\varphi^{(\nu)}(y). \end{aligned}$$

From Lemmas 4.1, 4.2 (a) and 4.3 we have

$$\begin{aligned} &\Big|P_{m_1-1-|\mu|}\Big(\frac{\partial^\mu}{\partial y^\mu}\Big(a(\cdot) - \sum_{|\alpha|=m_1-1}\frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})(\cdot)^\alpha\Big);\, y,\, z\Big)\Big| \\ &\quad\le C|y-z|^{m_1-1-|\mu|}\sum_{|\eta|=m_1-1}\Big(\frac{1}{|Q(y,\, z)|}\int_{Q(y,\, z)} |a^{(\eta)}(t) - m_{Q_j}(a^{(\eta)})|^q dt\Big)^{1/q} \\ &\quad\le C|y-z|^{m_1-1-|\mu|}\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}|\bar{\bar{Q}}_j|^{\beta/n}. \end{aligned}$$

From the assumptions on φ, we have $|\varphi^{(\nu)}(y)| \le C|y-z|^{-|\nu|}$. So for $|\gamma| = m_1 - 1$ we have $|a_\varphi^{(\gamma)}(y)| \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}|\bar{\bar{Q}}_j|^{\beta/n}$. Since a_φ has support in $\bar{\bar{Q}}_j$, we finally obtain, for $|\gamma| = m_1 - 1$,

$$\|a_\varphi^{(\gamma)}\|_q \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}|\bar{\bar{Q}}_j|^{\beta/n+1/q}. \tag{8}$$

Similarly, for $|\gamma| = m_2 - 1$ we have the estimate

$$\|b_\varphi^{(\gamma)}\|_q \le C\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}|\bar{\bar{Q}}_j|^{\beta/n+1/q}. \tag{9}$$

We observe that for $y \in \bar{Q}_j$,

$$\begin{aligned} P_{m_1}(a;\, x,\, y) &= P_{m_1}\Big(P_{m_1-1}(a;\, (\cdot),\, z)\varphi(\cdot);\, x,\, y\Big) \\ &= P_{m_1}\Big(P_{m_1-1}\Big(a(t) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})t^\alpha;\, (\cdot),\, z\Big)\varphi(\cdot);\, x,\, y\Big) \\ &= P_{m_1}(a_\varphi;\, x,\, y), \end{aligned} \tag{10}$$

and likewise, $P_{m_2}(b;\, x,\, y) = P_{m_2}(b_\varphi;\, x,\, y)$. It now follows that for $x \in Q_j$,

$$C_*(a,\, b;\, f_1)(x) = C_*(a_\varphi,\, b_\varphi;\, f_1)(x).$$

By Cohen and Gosselin's [3, p. 451] result, for $1/r = 1/p + 2/q < 1$ we have

$$\begin{aligned} &\Big|\{x \in Q_j : C_*(a,\, b;\, f_1)(x) > \delta\lambda\}\Big| \\ &\quad= \Big|\{x \in Q_j : C_*(a_\varphi,\, b_\varphi;\, f_1)(x) > \delta\lambda\}\Big| \\ &\quad\le C\Big[\frac{\|C_*(a_\varphi,\, b_\varphi;\, f_1)\|_r}{\delta\lambda}\Big]^r \\ &\quad\le C(\delta\lambda)^{-r}\Big(\Big(\sum_{|\alpha|=m_1-1} \|a_\varphi^{(\alpha)}\|_q\Big)\Big(\sum_{|\eta|=m_2-1} \|b_\varphi^{(\eta)}\|_q\Big)\|f_1\|_p\Big)^r \\ &\quad\le C(\delta\lambda)^{-r}\Big(\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta,\,p}(f)(z)\Big)^r |\bar{\bar{Q}}_j|^{r(2/q+1/p)} \\ &\quad\le C(\delta\lambda)^{-r}(\gamma\lambda)^r|\bar{\bar{Q}}_j| \\ &\quad\le C\Big(\frac{\gamma}{\delta}\Big)^r|Q_j|. \end{aligned} \tag{11}$$

This completes the estimate on f_1.

The f_2 estimate for $\epsilon \approx \operatorname{diam}(\tilde{Q}_j)$. Let $K = K(n)$ be a large positive integer depending only on n. The estimation for f_2 is split into the two cases $\operatorname{diam}(\tilde{Q}_j) \le \epsilon \le K(n)\operatorname{diam}(\tilde{Q}_j)$ and $\epsilon > K(n)\operatorname{diam}(\tilde{Q}_j)$. The case of $\epsilon < \operatorname{diam}(\tilde{Q}_j)$ is ignored since f_2 has support outside $\bar{Q}_j$. Let

$$a_j(x) = a(x) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})x^\alpha \quad\text{and}\quad b_j(x) = b(x) - \sum_{|\eta|=m_2-1} \frac{1}{\eta!} m_{Q_j}(b^{(\eta)})x^\eta.$$

We now set $k(x,\, y) = P_{m_1}(a_j;\, x,\, y)P_{m_2}(b_j;\, x,\, y)\frac{\Omega(x-y)}{|x-y|^{n+M-2}}$.

We now choose x_0 in $\tilde{Q}_j$ with $x_0 \in \mathbb{R}^n \setminus \cup Q_j$, and for $x \in Q_j$

$$\begin{aligned} |C_\epsilon(a,\, b;\, f_2)(x)| &= |C_\epsilon(a_j,\, b_j;\, f_2)(x)| \\ &\le \Big|\int_{|x-y|>\epsilon} [k(x,\, y) - k(x_0,\, y)]f_2(y)dy\Big| + \Big|\int_{\epsilon<|x-y|\le C\epsilon} k(x_0,\, y)f_2(y)dy\Big| \\ &\quad + \Big|\int_{\epsilon<|x_0-y|\le C\epsilon} k(x_0,\, y)f_2(y)\Big| + \Big|\int_{|x_0-y|>\epsilon} k(x_0,\, y)f_2(y)dy\Big|. \end{aligned} \tag{12}$$

In the last integral we write $f_2 = f - f_1$ and incorporate the f_1 part of this integral into the third integral after enlarging the region of integration to $\operatorname{diam}(\tilde{Q}_j) \le |x_0 - y| \le K(n)\operatorname{diam}(\tilde{Q}_j)$.

Here $K(n)$ is chosen to be large enough to ensure that the ball centered at x_0 with radius $K(n)\,\mathrm{diam}(\tilde{Q}_j)$ contains $\bar{Q}_j$. We finally obtain

$$\begin{aligned}\Big|C_\epsilon(a,\, b;\, f_2)(x)\Big| \le \Big|\int_{|x-y|>\epsilon}&[k(x,\, y)-k(x_0,\, y)]f_2(y)dy\Big|\\&+\int_{R(x)}|k(x_0,\, y)f(y)|dy+\int_{R(x_0)}|k(x_0,\, y)f(y)|dy\\&+|C_\epsilon(a,\, b;\, f)(x_0)|,\end{aligned}\tag{13}$$

where $R(\cdot)$ denotes the integral region $\mathrm{diam}(\tilde{Q}_j)\le|\cdot-y|\le K(n)\,\mathrm{diam}(\tilde{Q}_j)$. The last integral is bounded by λ since $x_0\notin\cup Q_j$. The middle integrals are error terms which we will estimate later. We now deal with the first integral. We have

$$\begin{aligned}&k(x,\, y)-k(x_0,\, y)\\&\quad=P_{m_1}(a_j;\, x,\, y)P_{m_2}(b_j;\, x,\, y)\Big[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big]\\&\qquad+\Big[P_{m_1}(a_j;\, x,\, y)-P_{m_1}(a_j;\, x_0,\, y)\Big]P_{m_2}(b_j;\, x,\, y)\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\\&\qquad+P_{m_1}(a_j;\, x_0,\, y)\Big[P_{m_2}(b_j;\, x,\, y)-P_{m_2}(b_j;\, x_0,\, y)\Big]\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\\&\quad\equiv k_1(x,\, x_0,\, y)+k_2(x,\, x_0,\, y)+k_3(x,\, x_0,\, y).\end{aligned}\tag{14}$$

For $|x-y|>\epsilon$, standard arguments imply

$$\Big|\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big|\le C\frac{|x-x_0|}{|x-y|^{n+M-1}}.\tag{15}$$

We now write

$$\Big|\int_{|x-y|>\epsilon}k_1(x,\, x_0,\, y)f_2(y)dy\Big|\le\sum_{\mu=1}^{4}\Big|\int_{|x-y|>\epsilon}k_{1\mu}(x,\, x_0,\, y)f_2(y)dy\Big|,\tag{16}$$

where

$$\begin{aligned}k_{11}(x,\, x_0,\, y)&=P_{m_1-1}(a_j;\, x,\, y)P_{m_2-1}(b_j;\, x,\, y)\Big[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big],\\k_{12}(x,\, x_0,\, y)&=\Big(\sum_{|\alpha|=m_1-1}\frac{(x-y)^\alpha}{\alpha!}a_j^{(\alpha)}(y)\Big)P_{m_2-1}(b_j;\, x,\, y)\\&\qquad\times\Big[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big],\\k_{13}(x,\, x_0,\, y)&=P_{m_1-1}(a_j;\, x,\, y)\Big(\sum_{|\eta|=m_2-1}\frac{(x-y)^\eta}{\eta!}b_j^{(\eta)}(y)\Big)\\&\qquad\times\Big[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big],\\k_{14}(x,\, x_0,\, y)&=\Big(\sum_{|\alpha|=m_1-1}\frac{(x-y)^\alpha}{\alpha!}a_j^{(\alpha)}(y)\Big)\Big(\sum_{|\eta|=m_2-1}\frac{(x-y)^\eta}{\eta!}b_j^{(\eta)}(y)\Big)\\&\qquad\times\Big[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\Big].\end{aligned}$$

We estimate these integrals separately. We have

$$
\begin{aligned}
&\Big|\int_{|x-y|>\epsilon} k_{11}(x,\,x_0,\,y) f_2(y)dy\Big| \\
&\le C\operatorname{diam}(\tilde Q_j)\int_{|x-y|>\epsilon}\Big|\frac{P_{m_1-1}(a_j;\,x,\,y)}{|x-y|^{m_1-1}}\Big|\Big|\frac{P_{m_2-1}(b_j;\,x,\,y)}{|x-y|^{m_2-1}}\Big|\frac{|f(y)|}{|x-y|^{n+1}}dy \\
&\le C\operatorname{diam}(\tilde Q_j)\int_{|x-y|>\epsilon}\left(\sum_{|\alpha|=m_1-1}\Big(\frac{1}{|Q(x,\,y)|}\int_{Q(x,\,y)}|a^{(\alpha)}(t)-m_{Q_j}(a^{(\alpha)})|^q dt\Big)^{1/q}\right) \\
&\quad\times\left(\sum_{|\eta|=m_2-1}\Big(\frac{1}{|Q(x,\,y)|}\int_{Q(x,\,y)}|b^{(\eta)}(t)-m_{Q_j}(b^{(\eta)})|^q dt\Big)^{1/q}\right)\frac{|f(y)|}{|x-y|^{n+1}}dy.
\end{aligned}
$$

Replacing $m_{Q_j}(a^{(\alpha)})$ by $m_{Q(x,\,y)}(a^{(\alpha)})$ and likewise for $b^{(\eta)}$ and then using an estimate from [8, Lemma 2.4] we obtain

$$
\begin{aligned}
\Big|\int_{|x-y|>\epsilon} k_{11}(x,\,x_0,\,y) f_2(y)dy\Big| \;&\le\; C\|\nabla^{m_1-1}a\|_{\dot\Lambda_\beta}\|\nabla^{m_2-1}b\|_{\dot\Lambda_\beta}\operatorname{diam}(\tilde Q_j)|\tilde Q_j|^{2\beta/n} \\
&\quad\times\int_{|x-y|>\epsilon}\frac{\Big(1+\log\Big(\frac{|x-y|}{\operatorname{diam}(\tilde Q_j)}\Big)\Big)^2|f(y)|}{|x-y|^{n+1}}dy.
\end{aligned}
$$

The last integral can be estimated as

$$
\begin{aligned}
&\operatorname{diam}(\tilde Q_j)|\tilde Q_j|^{2\beta/n}\sum_{\nu=0}^{\infty}\int_{2^\nu\epsilon<|x-y|\le 2^{\nu+1}\epsilon}\frac{\Big(1+\log\Big(\frac{|x-y|}{\operatorname{diam}(\tilde Q_j)}\Big)\Big)^2|f(y)|}{|x-y|^{n+1}}dy \\
&\le\sum_{\nu=0}^{\infty}(\nu+2)^2|\tilde Q_j|^{2\beta/n}\frac{\operatorname{diam}(\tilde Q_j)}{(2^\nu\epsilon)^{n+1}}\int_{|x-y|<2^{\nu+1}\epsilon}|f(y)|dy \\
&\le C\sum_{\nu=0}^{\infty}\frac{(\nu+2)^2}{2^{\nu(1+2\beta)}}(2^{\nu+1}\epsilon)^{2\beta}\Big(\frac{1}{(2^{\nu+1}\epsilon)^n}\int_{|x-y|<2^{\nu+1}\epsilon}|f(y)|^p dy\Big)^{1/p} \\
&\le CM_{2\beta,\,p}(f)(z).
\end{aligned}
$$

Thus we obtain

$$
\Big|\int_{|x-y|>\epsilon} k_{11}(x,\,x_0,\,y) f_2(y)dy\Big|\le C\|\nabla^{m_1-1}a\|_{\dot\Lambda_\beta}\|\nabla^{m_2-1}b\|_{\dot\Lambda_\beta}M_{2\beta,\,p}(f)(z)\le C\gamma\lambda. \tag{17}
$$

For the next integral we have

$$
\begin{aligned}
&\Big|\int_{|x-y|>\epsilon} k_{12}(x,\,x_0,\,y) f_2(y)dy\Big| \\
&\le\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\int_{|x-y|>\epsilon}\frac{|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})||P_{m_2-1}(b_j;\,x,\,y)||x-x_0||f(y)|dy}{|x-y|^{n+m_2}} \\
&\le\sum_{|\alpha|=m_1-1}\frac{\operatorname{diam}(\tilde Q_j)}{\alpha!}\int_{|x-y|>\epsilon}\Big|\frac{P_{m_2-1}(b_j;\,x,\,y)}{|x-y|^{m_2-1}}\Big|\frac{|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})|\,|f(y)|}{|x-y|^{n+1}}dy \\
&\le\sum_{|\alpha|=m_1-1}\frac{\operatorname{diam}(\tilde Q_j)}{\alpha!}\|\nabla^{m_2-1}b\|_{\dot\Lambda_\beta}|\tilde Q_j|^{\beta/n}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{\nu=0}^{\infty} \int_{2^\nu\epsilon<|x-y|\le 2^{\nu+1}\epsilon} \frac{\Big(1+\log\frac{|x-y|}{\operatorname{diam}(\tilde{Q}_j)}\Big)|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})|\,|f(y)|dy}{|x-y|^{n+1}}\\
&\le C\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\sum_{\nu=0}^{\infty}\frac{(\nu+2)}{2^{\nu(1+\beta)}}(2^{\nu+1}\epsilon)^{\beta}\\
&\quad\times\Big(\frac{1}{(2^{\nu+1}\epsilon)^n}\int_{|x-y|<2^{\nu+1}\epsilon}|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})|\,|f(y)|dy\Big)\\
&\le C\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\sum_{\nu=0}^{\infty}\frac{(\nu+2)}{2^{\nu(1+\beta)}}\\
&\quad\times(2^{\nu+1}\epsilon)^{-\beta}\Big(\frac{1}{(2^{\nu+1}\epsilon)^n}\int_{|x-y|<2^{\nu+1}\epsilon}|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})|^{p'}dy\Big)^{1/p'}\\
&\quad\times(2^{\nu+1}\epsilon)^{2\beta}\Big(\frac{1}{(2^{\nu+1}\epsilon)^n}\int_{|x-y|<2^{\nu+1}\epsilon}|f(y)|^{p}dy\Big)^{1/p}\\
&\le C\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}M_{2\beta,\,p}(f)(z)\sum_{\nu=0}^{\infty}\frac{\nu(\nu+2)}{2^{\nu(1+\beta)}}|2^{\nu+1}\tilde{Q}_j|^{-\beta/n}\\
&\quad\times\Big(\frac{1}{|2^{\nu+1}\tilde{Q}_j|}\int_{2^{\nu+1}\tilde{Q}_j}|a^{(\alpha)}(y)-m_{2^{\nu+1}\tilde{Q}_j}(a^{(\alpha)})|^{p'}dy\Big)^{1/p'}\\
&\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}M_{2\beta,\,p}(f)(z)\\
&\le C\gamma\lambda. \qquad (18)
\end{aligned}
$$

A completely similar argument will give us the same estimate for $|\int_{|x-y|>\epsilon}k_{13}(x,\,x_0,\,y)f_2(y)dy|$. Finally, we note that the same estimate for $|\int_{|x-y|>\epsilon}k_{14}(x,\,x_0,\,y)f_2(y)dy|$ follows from an argument analogous to the above except that both a and b terms must be treated in the way the a terms were treated above. The details are omitted here.

Returning to (14), we deal now with the second term involving $k_2(x,\,x_0,\,y)$. A similar argument will apply to the last term involving $k_3(x,\,x_0,\,y)$. We use the following identity (see [9])

$$
\begin{aligned}
&P_{m_1}(a_j;\,x,\,y)-P_{m_1}(a_j;\,x_0,\,y)\\
&\quad=P_{m_1}(a_j;\,x,\,x_0)+\sum_{0<|\alpha|<m_1}\frac{(x-x_0)^\alpha}{\alpha!}P_{m_1-|\alpha|}(a_j^{(\alpha)};\,x_0,\,y)\\
&\quad=P_{m_1-1}(a_j;\,x,\,x_0)+\sum_{0<|\alpha|<m_1-1}\frac{(x-x_0)^\alpha}{\alpha!}P_{m_1-|\alpha|}(a_j^{(\alpha)};\,x_0,\,y)\\
&\qquad-\sum_{|\alpha|=m_1-1}\frac{(x-x_0)^\alpha}{\alpha!}a_j^{(\alpha)}(y). \qquad (19)
\end{aligned}
$$

Each of the three terms above can now be used to write $k_2(x,\,x_0,\,y)$ as

$$k_{21}(x,\,x_0,\,y)+k_{22}(x,\,x_0,\,y)+k_{23}(x,\,x_0,\,y).$$

We now estimate each of the corresponding integrals separately. We have

$$\begin{aligned}
&\Big|\int_{|x-y|>\epsilon} k_{21}(x,\, x_0,\, y) f_2(y)dy\Big| \\
&\quad = \Big|\int_{|x-y|>\epsilon} P_{m_1-1}(a_j;\, x,\, x_0) P_{m_2}(b_j;\, x,\, y) \frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} f_2(y)dy\Big| \\
&\quad \le C\Big|\int_{|x-y|>\epsilon} P_{m_1-1}(a_j;\, x,\, x_0) P_{m_2-1}(b_j;\, x,\, y) \frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} f_2(y)dy\Big| \\
&\qquad + C \sum_{|\eta|=m_2-1} \frac{1}{\eta!} \\
&\qquad\quad \times \Big|\int_{|x-y|>\epsilon} P_{m_1-1}(a_j;\, x,\, x_0) b_j^{(\eta)}(y)(x-y)^{\eta} \frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} f_2(y)dy\Big|. \qquad (20)
\end{aligned}$$

The first integral in (20) is majorized by

$$\begin{aligned}
&C|x-x_0| \int_{|x-y|>\epsilon} \Big|\frac{P_{m_1-1}(a_j;\, x,\, x_0)}{|x-x_0||x-y|^{m_1-2}}\Big|\Big|\frac{P_{m_2-1}(b_j;\, x,\, y)}{|x-y|^{m_2-1}}\Big| \frac{|f(y)|dy}{|x-y|^{n+1}} \\
&\quad \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta,\, p}(f)(z) \le C\gamma\lambda. \qquad (21)
\end{aligned}$$

Each integral in the sum in (20) is majorized by

$$\begin{aligned}
&C|x-x_0| \int_{|x-y|>\epsilon} \Big|\frac{P_{m_1-1}(a_j;\, x,\, x_0)}{|x-x_0||x-y|^{m_1-2}}\Big| \frac{|b_j^{(\eta)}(y)||f(y)|dy}{|x-y|^{n+1}} \\
&\quad \le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta,\, p}(f)(z) \le C\gamma\lambda. \qquad (22)
\end{aligned}$$

We note that each of the above estimates follows from the same type of argument used in estimating the integral with k_{12} above. We now estimate the integral with $k_{22}(x,\, x_0,\, y)$. We have

$$\begin{aligned}
&\Big|\int_{|x-y|>\epsilon} k_{22}(x,\, x_0,\, y) f_2(y)dy\Big| \\
&\quad \le C \sum_{0<|\alpha|<m_1-1} \frac{1}{\alpha!} \\
&\qquad \times \Big|\int_{|x-y|>\epsilon} P_{m_1-|\alpha|}(a_j^{(\alpha)};\, x_0,\, y) P_{m_2-1}(b_j;\, x,\, y) \frac{(x-x_0)^{\alpha}\Omega(x_0-y)}{|x_0-y|^{n+M-2}} f_2(y)dy\Big| \\
&\quad + C \sum_{0<|\alpha|<m_1-1} \sum_{|\eta|=m_2-1} \frac{1}{\alpha!\eta!} \\
&\qquad \times \Big|\int_{|x-y|>\epsilon} P_{m_1-|\alpha|}(a_j^{(\alpha)};\, x_0,\, y) \frac{(x-x_0)^{\alpha}(x-y)^{\eta}\Omega(x_0-y)}{|x_0-y|^{n+M-2}} b_j^{(\eta)}(y) f_2(y)dy\Big| \\
&\quad \equiv \mathrm{I}'_{22} + \mathrm{I}''_{22}. \qquad (23)
\end{aligned}$$

We now estimate I'_{22} by breaking up the first remainder. We have

$$\begin{aligned}
\mathrm{I}'_{22} &\le C \sum_{0<|\alpha|<m_1-1} \frac{1}{\alpha!}|x-x_0| \\
&\quad \times \int_{|x-y|>\epsilon} \Big|\frac{P_{m_1-|\alpha|-1}(a_j^{(\alpha)};\, x_0,\, y)}{|x_0-y|^{m_1-|\alpha|-1}}\Big|\Big|\frac{P_{m_2-1}(b_j;\, x,\, y)}{|x-y|^{m_2-1}}\Big| \frac{|f(y)|dy}{|x-y|^{n+1}}
\end{aligned}$$

$$+ C \sum_{0<|\alpha|<m_1-1} \frac{1}{\alpha!} \sum_{|\gamma|=m_1-|\alpha|-1} \frac{1}{\gamma!} |x - x_0| \times \int_{|x-y|>\epsilon} \Big| \frac{P_{m_2-1}(b_j;\, x,\, y)}{|x-y|^{m_2-1}} \Big| \frac{|a_j^{(\alpha+\gamma)}(y)||f(y)|dy}{|x-y|^{n+1}}. \tag{24}$$

By familiar arguments, each of these integrals is bounded by

$$\|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} M_{2\beta,\,p}(f)(z) \le C\gamma\lambda.$$

We now estimate I''_{22} again by breaking up the first remainder. We have

$$\begin{aligned} \mathrm{I}''_{22} \le{}& C \sum_{0<|\alpha|<m_1-1} \sum_{|\eta|=m_2-1} \frac{1}{\alpha!\eta!} |x-x_0| \\ &\times \int_{|x-y|>\epsilon} \Big| \frac{P_{m_1-|\alpha|-1}(a_j^{(\alpha)};\, x_0,\, y)}{|x_0-y|^{m_1-|\alpha|-1}} \Big| \frac{|b_j^{(\eta)}(y)||f(y)|dy}{|x-y|^{n+1}} \\ &+ C \sum_{0<|\alpha|<m_1-1} \sum_{|\eta|=m_2-1} \frac{1}{\alpha!\eta!} \sum_{|\gamma|=m_1-|\alpha|-1} \frac{1}{\gamma!} |x-x_0| \\ &\times \int_{|x-y|>\epsilon} \frac{|a_j^{(\alpha+\gamma)}(y)|\,|b_j^{(\eta)}(y)|\,|f(y)|}{|x-y|^{n+1}}. \end{aligned} \tag{25}$$

Again by familiar arguments each of these integrals is bounded by

$$C\|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} M_{2\beta,\,p}(f)(z) \le C\gamma\lambda.$$

This completes the estimate of the integral in (23) involving $k_{22}(x,\, x_0,\, y)$. The same estimate holds for the integral involving $k_{23}(x,\, x_0,\, y)$ by the above argument with the roles of a and b being interchanged. This now completes the estimate of the first term in (13). To summarize, we have shown that

$$\Big| \int_{|x-y|>\epsilon} [k(x,\, y) - k(x_0,\, y)] f_2(y) dy \Big| \le C\gamma\lambda.$$

To complete the f_2 estimate for $\operatorname{diam}(\tilde{Q}_j) \le \epsilon < K(n) \operatorname{diam}(\tilde{Q}_j)$, it only remains to estimate the error terms in (13). We will estimate the first error term while the second one is handled similarly. We have

$$\begin{aligned} &\int_{R(x)} |k(x_0,\, y) f(y)| dy \\ &= \int_{R(x)} |P_{m_1}(a_j;\, x_0,\, y) P_{m_2}(b_j;\, x_0,\, y)| \Big| \frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} \Big| |f(y)| dy \\ &\le \int_{R(x)} \Big| \frac{P_{m_1-1}(a_j;\, x_0,\, y)}{|x_0-y|^{m_1-1}} \Big| \Big| \frac{P_{m_2-1}(b_j;\, x_0,\, y)}{|x_0-y|^{m_2-1}} \Big| \frac{|f(y)|}{|x-y|^{n+1}} dy \\ &\quad + \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} \int_{R(x)} \Big| \frac{P_{m_2-1}(b_j;\, x_0,\, y)}{|x_0-y|^{m_2-1}} \Big| \frac{|a_j^{(\alpha)}(y)||f(y)|}{|x-y|^{n}} dy \end{aligned}$$

$$+\sum_{|\eta|=m_2-1}\frac{1}{\eta!}\int_{R(x)}\Big|\frac{P_{m_1-1}(a_j;\,x_0,\,y)}{|x_0-y|^{m_1-1}}\Big|\frac{|b_j^{(\eta)}(y)||f(y)|}{|x-y|^n}dy$$
$$+\sum_{|\alpha|=m_1-1}\sum_{|\eta|=m_2-1}\frac{1}{\alpha!\eta!}\int_{R(x)}\frac{|a_j^{(\alpha)}(y)||b_j^{(\eta)}(y)||f(y)|}{|x-y|^n}dy. \qquad (26)$$

We note that for $y\in R(x)$, $1\le\frac{K(n)\operatorname{diam}(\tilde{Q}_j)}{|x-y|}$. Thus each error term is increased by multiplying by $K(n)\operatorname{diam}(\tilde{Q}_j)$ on the outside and increasing the exponent of $|x-y|$ under $|f(y)|$ by 1. After doing so each integral can be estimated by familiar arguments used earlier. Each error term is majorized by

$$CK(n)\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}M_{2\beta,\,p}(f)(z).$$

It now follows that the entire error term is majorized by $C(n)\gamma\lambda$. In summary we have shown that for any $x\in Q_j$, $\sup_{\epsilon\approx\operatorname{diam}(\tilde{Q}_j)}|C_\epsilon(a,\,b;\,f_2)(x)|\le C\gamma\lambda+\lambda$.

The f_2 estimate for $\epsilon>K(n)\operatorname{diam}(\tilde{Q}_j)$. Let Q_j^ϵ denote the cube with sides parallel to the axes with the same center as Q_j and diameter ϵ. Let $a_\epsilon(x)=a(x)-\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}m_{Q_j^\epsilon}(a^{(\alpha)})x^\alpha$ and let $b_\epsilon(x)$ be defined in a similar manner. Let $k_\epsilon(x,y)=P_{m_1}(a_\epsilon;x,y)P_{m_2}(b_\epsilon;x,y)\frac{\Omega(x-y)}{|x-y|^{n+M-2}}$. Proceeding as before we have

$$|C_\epsilon(a,\,b;\,f_2)(x)|\le\Big|\int_{|x-y|>\epsilon}[k_\epsilon(x,\,y)-k_\epsilon(x_0,\,y)]f_2(y)dy\Big|+\int_{\epsilon\le|x-y|\le C\epsilon}|k_\epsilon(x_0,\,y)f(y)|dy$$
$$+\int_{\epsilon\le|x_0-y|\le C\epsilon}|k_\epsilon(x_0,\,y)f(y)|dy+\Big|\int_{|x_0-y|>\epsilon}k_\epsilon(x_0,\,y)f(y)dy\Big|. \qquad (27)$$

The last integral in (27) is bounded by λ since $x_0\notin\cup_jQ_j$. The same estimates hold for the error terms since $\operatorname{diam}(Q_j^\epsilon)=\epsilon$. For the first term we must be careful since $|x-x_0|$ and $\operatorname{diam}(Q_j^\epsilon)$ are no longer comparable. To deal with such terms we select a point x_ϵ such that $|x-x_\epsilon|=2\epsilon$. Then $\epsilon<|x-x_0|<3\epsilon$. We then write

$$P_{m_1-1}(a_\epsilon;\,x,\,x_0)=P_{m_1-1}(a_\epsilon;\,x,\,x_\epsilon)-P_{m_1-1}(a_\epsilon;\,x_0,\,x_\epsilon)$$
$$-\sum_{0<|\alpha|<m_1-1}\frac{(x-x_0)^\alpha}{\alpha!}P_{m_1-1-|\alpha|}(a_\epsilon^{(\alpha)};\,x_0,\,x_\epsilon). \qquad (28)$$

The appropriate estimates now hold for each of these terms. For example, the first term in the first integral in (20) can be estimated as follows.

$$\Big|\int_{|x-y|>\epsilon}P_{m_1-1}(a_\epsilon;\,x,\,x_\epsilon)P_{m_2-1}(b_\epsilon;\,x,\,y)\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}f_2(y)dy\Big|$$
$$\le C|x-x_\epsilon|\int_{|x-y|>\epsilon}\Big|\frac{P_{m_1-1}(a_\epsilon;\,x,\,x_\epsilon)}{|x-x_\epsilon|^{m_1-1}}\Big|\Big|\frac{P_{m_2-1}(b_\epsilon;\,x,\,y)}{|x-y|^{m_2-1}}\Big|\frac{|f(y)|}{|x-y|^{n+1}}dy$$
$$\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}|Q_j^\epsilon|^{\beta/n}\Big(\epsilon\int_{|x-y|>\epsilon}\Big|\frac{P_{m_2-1}(b_\epsilon;\,x,\,y)}{|x-y|^{m_2-1}}\Big|\frac{|f(y)|}{|x-y|^{n+1}}dy\Big)$$
$$\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}M_{2\beta,\,p}(f)(z)\le C\gamma\lambda. \qquad (29)$$

All the remaining estimates can be handled similarly. Ultimately, for $\epsilon>K(n)\operatorname{diam}(\tilde{Q}_j)$, we obtain

$$\Big|\int_{|x-y|>\epsilon}[k_\epsilon(x,\,y)-k_\epsilon(x_0,\,y)]f_2(y)dy\Big|\le C\gamma\lambda. \qquad (30)$$

From (27) it now follows that

$$\sup_{\epsilon > K(n)\,\mathrm{diam}(\tilde{Q}_j)} |C_\epsilon(a,\, b;\, f_2)(x)| \le \lambda + C\gamma\lambda. \tag{31}$$

The estimates on f_2 now yield the pointwise estimate $C_*(a,\, b;\, f_2)(x) \le \lambda + C\gamma\lambda$ for all $x \in Q_j$.

We now choose γ_0 such that $C\gamma_0 < 1$, where C is the constant in (31). Then from (11) with $\delta = 1$, we have, for $\gamma < \gamma_0$,

$$\begin{aligned}
&\Big|\{x \in Q_j : C_*(a,\, b;\, f)(x) > 3\lambda,\ \|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta} M_{2\beta,\,p}(f)(x) \le \gamma\lambda\}\Big| \\
&\quad \le \Big|\{x \in Q_j : C_*(a,\, b;\, f_1)(x) > 2\lambda - C\gamma\lambda\}\Big| \\
&\qquad + \Big|\{x \in Q_j : C_*(a,\, b;\, f_2)(x) > \lambda + C\gamma\lambda\}\Big| \\
&\quad \le \Big|\{x \in Q_j : C_*(a,\, b;\, f_1)(x) > \lambda\}\Big| \\
&\quad \le C\gamma^r |Q_j|.
\end{aligned} \tag{32}$$

This establishes the good λ inequality and completes the proof of Theorem 3.1.

Theorem 3.2 now follows from the good λ inequality and Lemma 2.1 by a standard good λ argument.

We now turn to the proof of Theorem 3.3. For a cube Q and $f \in L^p(\mathbb{R}^n)$, we decompose $f = f_1 + f_2$ with $f_1 = f\chi_{\bar{Q}}$. We fix a cube Q and $x \in Q$. Let x_0 be a point on the boundary of $\bar{\bar{Q}}$. We similarly define

$$\begin{aligned}
a_\varphi(y) &\equiv P_{m_1-1}\Big(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_Q(a^{(\alpha)});\, y,\, x_0\Big)\varphi(y), \\
b_\varphi(y) &\equiv P_{m_2-1}\Big(b(\cdot) - \sum_{|\eta|=m_2-1} \frac{1}{\eta!} m_Q(b^{(\eta)});\, y,\, x_0\Big)\varphi(y),
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\bar{a}(y) &\equiv a(y) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_Q(a^{(\alpha)}) y^\alpha, \\
\bar{b}(y) &\equiv b(y) - \sum_{|\eta|=m_2-1} \frac{1}{\eta!} m_Q(b^{(\eta)}) y^\eta.
\end{aligned} \tag{34}$$

We now choose a point y_0 on the boundary of $\tilde{Q}$. We have

$$\begin{aligned}
&\frac{1}{|Q|^{1+2\beta/n}} \int_Q |C(a,\, b;\, f)(y) - m_Q(C(\bar{a},\, \bar{b};\, f))|dy \\
&\quad = \frac{1}{|Q|^{1+2\beta/n}} \int_Q |C(\bar{a},\, \bar{b};\, f(y) - m_Q(C(\bar{a},\, \bar{b};\, f))|dy \\
&\quad \le \frac{2}{|Q|^{1+2\beta/n}} \int_Q |C(\bar{a},\, \bar{b};\, f)(y) - C(\bar{a},\, \bar{b};\, f_2)(y_0)|dy \\
&\quad \le \frac{2}{|Q|^{1+2\beta/n}} \int_Q |C(\bar{a},\, \bar{b};\, f_1)(y)|dy \\
&\qquad + \frac{2}{|Q|^{1+2\beta/n}} \int_Q |C(\bar{a},\, \bar{b};\, f_2)(y) - C(\bar{a},\, \bar{b};\, f_2)(y_0)|dy \\
&\quad \equiv \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned} \tag{35}$$

To estimate the first term J_1, we use the fact that for $y \in Q$, $C(\bar{a}, \bar{b}; f_1)(y) = C(a_\varphi, b_\varphi; f_1)(y)$. If $1/r + 2/q = 1/s < 1$, then from the previous results, we have

$$\begin{aligned}
\mathrm{J}_1 &\le \frac{2}{|Q|^{1+2\beta/n}} \|C(a_\varphi, b_\varphi; f_1)\|_s |Q|^{1-1/s} \\
&\le \frac{2}{|Q|^{1+2\beta/n}} \|a_\varphi\|_q \|b_\varphi\|_q \|f_1\|_r |Q|^{1-1/s} \\
&\le \frac{C}{|Q|^{1+2\beta/n}} \|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} |Q|^{2\beta/n+2/q} \|f_1\|_r |Q|^{1-1/s} \\
&\le C \|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} \Big(\frac{1}{|Q|} \int_{\tilde{Q}} |f(y)|^r dy \Big)^{1/r} \\
&\le C \|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} M_r f(x). \qquad (36)
\end{aligned}$$

To get the estimate of J_2, let

$$k(y,z) = P_{m_1}(\bar{a}; y, z) P_{m_2}(\bar{b}; y, z) \frac{\Omega(y-z)}{|y-z|^{n+M-2}}. \qquad (37)$$

Then

$$C(\bar{a}, \bar{b}; f_2)(y) - C(\bar{a}, \bar{b}; f_2)(y_0) = \int_{\mathbb{R}^n} [k(y, z) - k(y_0, z)] f_2(z) dz. \qquad (38)$$

To compute the last integral, we decompose

$$\begin{aligned}
& k(y, z) - k(y_0, z) \\
&\quad = P_{m_1}(\bar{a}; y, z) P_{m_2}(\bar{b}; y, z) \Big[\frac{\Omega(y-z)}{|y-z|^{n+M-2}} - \frac{\Omega(y_0-z)}{|y_0-z|^{n+M-2}} \Big] \\
&\qquad + \Big[P_{m_1}(\bar{a}; y, z) - P_{m_1}(\bar{a}; y_0, z) \Big] P_{m_2}(\bar{b}; y, z) \frac{\Omega(y_0-z)}{|y_0-z|^{n+M-2}} \\
&\qquad + P_{m_1}(\bar{a}; y_0, z) \Big[P_{m_2}(\bar{b}; y, z) - P_{m_2}(\bar{b}; y_0, z) \Big] \frac{\Omega(y_0-z)}{|y_0-z|^{n+M-2}} \\
&\quad \equiv k_1(y, y_0, z) + k_2(y, y_0, z) + k_3(y, y_0, z). \qquad (39)
\end{aligned}$$

Proceeding as in the proof of Theorem 3.1, we can obtain

$$\Big| \int_{\mathbb{R}^n} [k(y, z) - k(y_0, z)] f_2(z) dz \Big| \le C |Q|^{2\beta/n} \Big(M f(x) + M_r f(x) \Big). \qquad (40)$$

For example, the integrals involving k_{11} and k_{12} can be estimated as follows.

$$\begin{aligned}
& \Big| \int_{\mathbb{R}^n} P_{m_1-1}(\bar{a}; y, z) P_{m_2-1}(\bar{b}; y, z) \Big[\frac{\Omega(y-z)}{|y-z|^{n+M-2}} - \frac{\Omega(y_0-z)}{|y_0-z|^{n+M-2}} \Big] f_2(z) dz \Big| \\
&\le \int_{\mathbb{R}^n} \Big| \frac{P_{m_1-1}(\bar{a}; y, z)}{|y-z|^{m_1-1}} \Big| \Big| \frac{P_{m_2-1}(\bar{b}; y, z)}{|y-z|^{m_2-1}} \Big| \frac{|y-y_0| |f_2(z)|}{|y-z|^{n+1}} dz \\
&\le C \|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} |Q|^{2\beta/n} \mathrm{diam}(Q) \\
&\quad \times \int_{|z-y| > C\mathrm{diam}(Q)} \frac{\Big(1 + \log \Big(\frac{|y-z|}{\mathrm{diam}(Q)} \Big) \Big)^2 |f(z)|}{|y-z|^{n+1}} dz \\
&\le C \|\nabla^{m_1-1} a\|_{\dot{\Lambda}_\beta} \|\nabla^{m_2-1} b\|_{\dot{\Lambda}_\beta} |Q|^{2\beta/n} M f(x) \qquad (41)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\Big(\sum_{|\alpha|=m_1-1}\frac{(y-z)^\alpha}{\alpha!}\bar{a}^{(\alpha)}(z)\Big)P_{m_2-1}(\bar{b};\,y,\,z)\\
&\quad\times\Big[\frac{\Omega(y-z)}{|y-z|^{n+M-2}}-\frac{\Omega(y_0-z)}{|y_0-z|^{n+M-2}}\Big]f_2(z)dz\Big|\\
&\le\sum_{|\alpha|=m_1-1}\frac{\mathrm{diam}(Q)}{\alpha!}\int_{|z-y|>C\mathrm{diam}(Q)}\Big|\frac{P_{m_2-1}(\bar{b};\,y,\,z)}{|y-z|^{m_2-1}}\Big|\frac{|\bar{a}^{(\alpha)}(z)||f(z)|}{|y-z|^{n+1}}dz\\
&\le C\sum_{|\alpha|=m_1-1}\frac{\mathrm{diam}(Q)}{\alpha!}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\\
&\quad\times\int_{|z-y|>C\mathrm{diam}(Q)}\Big(1+\log\frac{|y-z|}{\mathrm{diam}(Q)}\Big)\frac{|\bar{a}^{(\alpha)}(z)||f(z)|}{|y-z|^{n+1}}dz\\
&\le C\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}|Q|^{2\beta/n}\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\sum_{\nu=0}^{\infty}\frac{(\nu+2)}{2^{\nu(1-\beta)}}\Big(2^{\nu+1}C\mathrm{diam}(Q)\Big)^{-\beta}\\
&\quad\times\Big(\frac{1}{(2^{\nu+1}C\mathrm{diam}(Q))^n}\int_{|z-y|<2^{\nu+1}C\mathrm{diam}(Q)}|a^{(\alpha)}(z)-m_Q(a^{(\alpha)})|^{r'}dz\Big)^{1/r'}\\
&\quad\times\Big(\frac{1}{(2^{\nu+1}C\mathrm{diam}(Q))^n}\int_{|z-y|<2^{\nu+1}C\mathrm{diam}(Q)}|f(z)|^{r}dz\Big)^{1/r}\\
&\le C|Q|^{2\beta/n}\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}M_rf(x).
\end{aligned}\tag{42}$$

So,

$$\mathrm{J}_2\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\Big(Mf(x)+M_rf(x)\Big),\tag{43}$$

where $1<r<\infty$ can be sufficiently close to 1. In summary, we have shown that

$$\mathrm{J}_1+\mathrm{J}_2\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\Big(Mf(x)+M_rf(x)\Big).\tag{44}$$

Finally, using Lemma 4.2 (b), we obtain

$$\|C(a,\,b;\,f)\|_{\dot{F}_p^{2\beta,\,\infty}}\le C\|\nabla^{m_1-1}a\|_{\dot{\Lambda}_\beta}\|\nabla^{m_2-1}b\|_{\dot{\Lambda}_\beta}\|f\|_p.$$

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